IMPACT OF A DISK ON THE SURFACE OF AN IDEAL COMPRESSIBLE FLUID

(UDAR DISKA PO POVERKHNOSTI IDEAL'NOI SZHIMAEMOI ZHIDKOSTI)

PMM Vol.28, № 4, 1964, pp.797-800

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(Received April 23, 1964)

The axisymmetric problem of the impact of an absolutely rigid disk on the surface of an ideal compressible fluid is considered. This problem arises in connection with the entry problem of blunt bodies into a fluid, where the body experiences large overloads at the initial interval of time. In the present paper a solution of the problem for $0 \leqslant t < a / c_0$. is sought by the method of integral transforms. The axisymmetric solution obtained up to the second approximation inclusively is compared with the result of the exact solution of the planar problem [1 and 2].

1. Let the impact of an absolutely rigid disk of radius a with the free surface of an ideal compressible fluid, which occupies the lower half-space $z > 0$ (Fig.1), take place at the moment $t = 0$. It is assumed that the initial velocity of the disk is $v_0 \ll c_0$. Here c_0 is the velocity of sound of the undisturbed fluid. For these conditions, as is not difficult to show, the problem will be linear for the initial time interval $\Delta t \sim a/c_0$ where the compressibility of the fluid is substantial and is described in the cylindrical system of dimensionless coordinates r_1, z_1 by the following equation and conditions

$$\frac{\partial^2 \varphi}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial \varphi}{\partial r_1} + \frac{\partial^2 \varphi}{\partial z_1^2} = \frac{\partial^2 \varphi}{\partial \tau^2} \tag{1.1}$$

$$\varphi = 0 \quad \text{for } r_1 > 1, \quad z_1 = 0, \qquad \frac{\partial \varphi}{\partial z_1} = v(\tau)\, a \quad \text{for } 0 \leqslant r_1 < 1, \quad z_1 = 0$$

$$\varphi = \frac{\partial \varphi}{\partial \tau} = 0 \quad \text{for } \tau = 0, \qquad \tau = \frac{c_0 t}{a}, \qquad r_1 = \frac{r}{a}, \qquad z_1 = \frac{z}{a}$$

Here $\varphi(r_1, z_1, \tau)$ is the potential of the perturbed motion. In what follows the subscript of the dimensionless independent variables will be dropped.

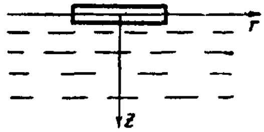

Fig. 1

Such a linear formulation gives the correct solution at all points with the exception of a small region at the edge of the disk where there must be a singularity because of the discontinuity in the direction of the velocities.

The force acting on the disk for $0 \leqslant \tau < 1$ is found in the problem. The analogous planar problem has been considered in [1 and 2].

2. The Laplace transform [3] with respect to τ is applied to the system (1.1)

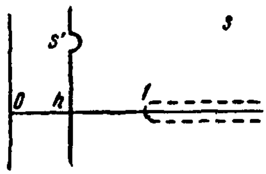

Fig. 2

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{\partial\Phi}{r\partial r}+\frac{\partial^2\Phi}{\partial z^2}=p^2\Phi \qquad (\mathrm{Re}\, p>0)$$

$$\Phi_{z=0}=0 \quad \text{for} \quad r>1 \qquad (2.1)$$

$$[\partial\Phi/\partial z]_{z=0}=V(p)\,a \quad \text{for} \quad 0\leqslant r<1$$

Then, applying the Hankel transform [3] with respect to r to Equation (2.1), it is not difficult to obtain for $z = 0$ (2.2)

$$\int_0^1 \Phi_{z=0} J_0(rx)\,r dr = -\frac{1}{\sqrt{p^2+x^2}}\left[\int_1^\infty \left(\frac{\partial\Phi}{\partial z}\right)_{z=0} J_0(rx)\,r dr + V(p)\,a\int_0^1 J_0(rx)\,r dr\right]$$

In (2.1) and (2.2) the following notation is adopted: $J_n(rx)$ is a Bessel function of nth order.

$$V(p) \doteqdot v(\tau), \qquad \Phi(r, z, p) \doteqdot \varphi(r, z, \tau)$$

The inverse Hankel transform for (2.2) with $\rho > 1$ gives (2.3)

$$\int_1^\infty \left(\frac{\partial\Phi}{\partial z}\right)_{z=0} r dr \int_0^\infty J_0(rx)\,J_0(\rho x)\frac{x\,dx}{\sqrt{p^2+x^2}} = -V(p)\,a\int_0^\infty J_1(x)\,J_0(x\rho)\frac{dx}{\sqrt{p^2+x^2}}$$

It is not difficult to show that

$$\int_0^\infty J_0(rx)\,J_0(x\rho)\frac{x\,dx}{\sqrt{p^2+x^2}} = \frac{1}{\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\alpha(s)\,s\,ds}{\sqrt{p^2-s^2}}$$

$$\int_0^\infty J_1(x)\,J_0(x\rho)\frac{dx}{\sqrt{p^2+x^2}} = \frac{1}{\pi i}\int_{b-i\infty}^{b+i\infty} K_0(s\rho)\,I_1(s)\frac{ds}{\sqrt{p^2-s^2}} \qquad (2.4)$$

Here $0<b<\mathrm{Re}\,p$; $K_n(x)$ and $I_n(x)$ are MacDonald and Bessel functions of imaginary argument corresponding to the nth order. And the branch of $\sqrt{p^2-s^2}$ is chosen so that $\sqrt{p^2-s^2}>0$ for $0<s<p$.

$$\alpha(s)=\begin{cases} K_0(sr)\,I_0(s\rho) & \text{for } r>\rho \\ K_0(s\rho)\,I_0(sr) & \text{for } \rho>r\end{cases}$$

Substituting Expressions (2.4) into (2.3) and making the variable change $s_1 = s/p$ (considering the simplicity $p > 0$), we obtain

$$\int_1^\infty \left(\frac{\partial\Phi}{\partial z}\right)_{z=0} r\,dr \int_{h-i\infty}^{h+i\infty}\frac{\alpha(ps)\,sds}{\sqrt{1-s^2}} = -\frac{a}{p}\,V(p)\int_{h-i\infty}^{h+i\infty} K_0(sp\rho)\,I_1(sp)\frac{ds}{\sqrt{1-s^2}} \qquad (2.5)$$

Here $0 < h < 1$ and the subscript 1 of s_1 has been omitted.

It is not difficult to notice now that the path of integration in (2.5) can be deformed along the branch cut $\sqrt{1-s^2}$, as shown in Fig.2, Then, having made one essential simplification, it is possible to apply asymptotic expansions of the cylindrical functions for large values of their arguments [4]. Let us represent

$$I_n(q) = {}^1/_2\,(-i)^n\,[H_n^{(2)}(qi)+H_n^{(1)}(qi)] \qquad (q = sp,\ spr,\ sp\rho)$$

Here $H_n^{(j)}(qi)$ is the Hankel function of jth kind and nth order.

It is not difficult to notice that the function $H_n^{(1)}(qi)$ gives a lag factor of $\exp(-2q)$ in comparison with $H_n^{(2)}(qi)$. Moreover, from the form of the deformed path it is seen that this factor satisfies the inequality $(-2q)\leqslant\exp(-2p)$.

Consequently, the function $H_n^{(1)}(qi)$ makes a contribution to the solution (2.5) in the form of secondary, tertiary, etc. diffraction waves.

For the inverse Laplace transform these terms appear only for $\tau > 2$, which is excluded by a condition of the problem. Consequently, in place of $I_1(q)$ in (2.5) it is necessary to substitute the function $\frac{1}{2}(-i)^n H_n^{(2)}(qi)$ and to make use of its asymptotics. It is not difficult to see that for the first approximation with respect to $1/p$ there is obtained (if the path is deformed into the previous position after applying the asymptotics)

$$\int_1^\infty \left(\frac{\partial\Phi}{\partial z}\right)_{z=0} \sqrt{r}\, dr \int_{h-i\infty}^{h+i\infty} \frac{e^{spr-sp\rho}}{\sqrt{1-s^2}}\, ds = -\frac{a}{p} V(p) \int_{h-i\infty}^{h+i\infty} \frac{e^{sp-sp\rho}}{s\sqrt{1-s^2}}\, ds \qquad (2.6)$$

Let us suppose that

$$\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} \sim M_0 (\rho - 1)^\beta \quad \text{for } \rho \to 1,\ \beta > -1$$

Then [5]

$$\psi(s, p)\, e^{-sp} \sim M_1 s^{-1-\beta} \qquad \text{for } s \to \infty$$

$$\left(\psi(s, p) = \int_1^\infty \left(\frac{\partial\Phi}{\partial z}\right)_{z=0} e^{spr} \sqrt{r}\, dr\right)$$

Applying the Wiener-Hopf method to Equation (2.6), we obtain

$$\int_{h-i\infty}^{h+i\infty} \left[\frac{\psi(s, p)}{\sqrt{1-s}} e^{-sp} + \frac{V(p)\, a}{ps\sqrt{1-s}}\right] \frac{ds}{s - s'} = 0 \qquad (2.7)$$

Making use of the fact that the left-hand term in the brackets of the integrand of (2.7) is an analytic function in the half-plane $\operatorname{Re} s < 1$, it is very easy to obtain from (2.7)

$$\psi(s, p) = V(p) \frac{a}{ps} (\sqrt{1-s} - 1) e^{sp} \qquad (2.8)$$

If the second approximation of $\psi(s,p)$ with respect to $1/p$ is sought, it is then necessary to take

$$\int_1^\infty \left(\frac{\partial\Phi}{\partial z}\right)_{z=0} \sqrt{r}\, dr \int_{h-i\infty}^{h+i\infty} \frac{e^{spr-sp\rho}}{\sqrt{1-s^2}}\, ds = -\frac{a}{p} V(p) \int_{h-i\infty}^{h+i\infty} \frac{e^{sp-sp\rho}}{s\sqrt{1-s^2}} \left(1 - \frac{3}{8ps} - \frac{1}{8ps\rho}\right) ds \qquad (2.9)$$

in place of (2.6).

It should be noted that the quantity γ

$$\left(\gamma = \frac{1}{8psr} - \frac{1}{8ps\rho}\right)$$

has been omitted in the term $1 + \gamma$ in the integrand of the left-hand side of (2.9).

This was done for the following reason. In order to find the term of second approximation, it is necessary to multiply both sides of the complete equation (2.9) by $\exp(s' p\rho)$ integrate with respect to ρ from 1 to infinity and determine the second term in the asymptotic expansion of the left-hand side in powers of $1/p$. It turns out that the quantity γ does not influence this term but is concerned with terms of higher order. This is easily shown if $[\partial\Phi / \partial z]_{z=0}$, obtained from (2.8) is substituted into the left-hand side of the complete equation (2.9). The result of the integration of the terms associated with γ then gives a quantity of third order in the expansion with respect to $1/p$.

Thus, after repeating for (2.9) all the arguments analogous to the case of the first approximation, we obtain

$$\psi(s, p) = e^{sp} a \frac{V(p)}{ps}\left[\sqrt{1-s} - 1 + \frac{1}{2sp}(1 - \sqrt{1-s}) - \frac{1}{4p}\sqrt{1-s}\right] \qquad (2.10)$$

From (2.10) it is not difficult to obtain

$$\int_1^\infty \left(\frac{\partial\Phi}{\partial z}\right)_{z=0} r dr = \frac{a}{p} V(p) \left[-\frac{1}{2} + \frac{1}{8p} + O(p^{-2})\right] \tag{2.11}$$

Then from (2.2) and (2.11) there is obtained

$$\int_0^1 \Phi_{z=0}\, r dr = \frac{a}{p} V(p) \left[-\frac{1}{2} + \frac{1}{2p} - \frac{1}{8p^2} + O(p^{-3})\right]$$

$$F(\tau) = -2\pi a \rho_0 c_0 \int_0^1 \left(\frac{\partial\varphi}{\partial\tau}\right)_{z=0} r dr \approx \pi a^2 \rho_0 c_0 \left[v(\tau) - \int_0^\tau v(x)\, dx + \frac{1}{4}\int_0^\tau v(x)(\tau - x)\, dx\right]$$

If the disk has a large mass, we can set $v(\tau) = v_0$ for $0 \leqslant \tau < 1$. Then to within $O(\tau^3)$ we obtain

$$F(\tau) = \pi a^2 \rho_0 c_0 v_0 (1 - \tau + {}^1/_8\tau^2) \tag{2.12}$$

From the planar problem [1] it is easy to obtain the force acting on the plate

$$P(\tau_1) = \rho_0 c_0 v_0 l (1 - {}^1/_2\tau_1) \tag{2.13}$$

$(\tau_1 = 2c_0 t / l)$ (l length of plate)

Graphs of the dimensionless functions

$$F_0(\tau) = \frac{F(\tau)}{\pi a^2 \rho_0 c_0 v_0}, \qquad P_0(\tau_1) = \frac{P(\tau_1)}{\rho_0 c_0 v_0 l}$$

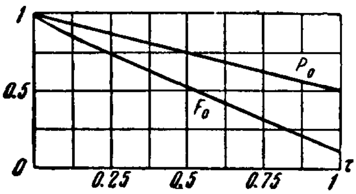

Fig. 3

are presented in Fig.3, where the subscript 1 of the variable τ_1 has been omitted.

From Fig.3 it can be concluded that even in the first approximation with respect to time the force acting on the disk is considerably smaller than the force acting on the plate. Since the first approximation with respect to time the solution of the axisymmetric problem must coincide with the planar problem, the result obtained for the forces is easily explained as the difference in integrating the same solution along the surface of the plate and the disk. This is easily verified, using the solution of the planar problem [1]. It is not even possible to obtain the term of second approximation from the solution of the planar problem.

To confirm the conclusions obtained, the solution of the axisymmetric problem of $[\partial\varphi / \partial z]_{z=0}$ up to the second approximation inclusively remains to be given and to be compared with the planar solution

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = \frac{v_0}{\pi\sqrt{r}}\left[\left(2 - \frac{\tau}{2}\right) \cos^{-1}\left(\frac{r-1}{\tau}\right)^{1/2} - 2\left(\frac{\tau}{r-1} - 1\right)^{1/2} - \frac{\tau - 4r + 4}{3\sqrt{r-1}} \times\right.$$

$$\left.\times \sqrt{\tau - r + 1} + \frac{1}{2}(\tau + 2r - 2) \tan^{-1}\left(\frac{\tau}{r-1} - 1\right)^{1/2}\right] \qquad \text{for } 1 < r < 1 + \tau$$

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = 0 \qquad \text{for } 1 + \tau < r$$

If $\tau \to 0$, in order that the quantity $(r - 1)/\tau$ be constant, only the first approximation which coincides with the solution of the problem [2] then remains

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} \sim \frac{2v_0}{\pi}\left[\cos^{-1}\left(\frac{r-1}{\tau}\right)^{1/2} - \left(\frac{\tau}{r-1} - 1\right)^{1/2}\right] \tag{2.14}$$

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Translated by R.D.C.